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The metric of moving bodies

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Abstract. The asymptotic form of the metric due to moving masses is calculated in terms of that due to a stationary mass. The work is modelled on the special-relativistic calculation of the electromagnetic field of moving charges from the Coulomb field of a stationary charge. The effects of acceleration and of terms not linear in the masses are neglected.

The Lense–Thirring metric for a slowly-rotating spherically symmetric body is derived. It is shown that the precession of spinning satellites, the deflection of starlight by the Sun, and the Shapiro time-delay are all determined by the same parameter in the metric.

The approximate metric due to an arbitrary moving mass distribution is shown to satisfy the linearized Einstein field equations.

1. Introduction

If one knows Coulomb's law (that is, the field due to a stationary charge), and if special relativity is valid, then one can calculate the electromagnetic field due to quite general current distributions. First one finds the field of an unaccelerated charge by Lorentz transformation of the Coulomb field. Then one argues that the same expression is approximately valid for a slowly accelerated charge, and that the field of an arbitrary set of such charges is the sum of their individual fields.

We shall use these results of special relativity as a model for a discussion of the gravitational field. The gravitational analogue of Coulomb's law is the expression for the metric due to a stationary, spherically symmetric body. We do not at first commit ourselves to any particular theory or field equations, but regard this metric as something to be determined empirically. There is no Lorentz covariance in a general spacetime, but we assume that spacetime is asymptotically flat in spatial directions, and we can then define a kind of asymptotic Lorentz covariance. This enables us to find an asymptotic expression for the metric due to a moving body. The contributions to the metric from different bodies are additive in the asymptotic region, and one can therefore calculate the asymptotic form of the metric due to a quite general mass distribution. In particular, for a spherically symmetric slowly-rotating body, the metric is of the familiar non-diagonal Lense–Thirring form. A slightly surprising result of the analysis is that the measurement of the deflection of starlight by the Sun, the Shapiro time-delay experiment, and the proposed measurement of the precession of a spinning satellite, should give the same information about the metric.

In the last part of the paper we show that the approximate metric we have calculated satisfies the linearized Einstein field equations.

2. Generalized Minkowski charts

We are going to consider only spacetimes on which a certain kind of coordinate system can be defined. In these coordinate systems, which are called *generalized Minkowski charts*, the metric tends asymptotically to the Minkowski form at spatial infinity. We begin by making these vague statements more precise.

Assume that spacetime Σ is geodesically complete. Let Γ be the set of all spacelike geodesics, $F\colon \Sigma \to R^1$ be a function, and $n \in R^1$. Then F is said to be $O(L^n)$ (and we write $F = O(L^n)$) if $|F(\gamma(u))u^{-n}|$ is bounded as $u \to \pm\infty$ for all $\gamma \in \Gamma$. If the coordinates of the point p are $x = (x^0, x^1, x^2, x^3)$ in some chart, and if $f(x) = F(p)$ for all $p \in \Sigma$, then we also write $f = O(L^n)$.

A chart $S\colon \Sigma \to R^4$ with coordinates x is defined to be a *generalized Minkowski chart* if the components $g_{\mu\nu}$ of the metric in S are given by

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{2.1}$$

where

$$\eta_{mn} = \delta_{mn}, \qquad \eta_{\mu 0} = \eta_{0\mu} = -\delta_{\mu 0}, h_{\mu\nu} = O(L^{-1}).$$

(Lower case Latin and Greek indices have the ranges $\{1, 2, 3\}$ and $\{0, 1, 2, 3\}$ respectively, and the summation convention applies to these indices only). The condition that the domain of S be the whole of Σ is convenient, but probably not essential.

We assume that the conditions we have just imposed on the metric can be satisfied in some chart S if the sources of the gravitational field and all gravitational radiation are confined to a bounded region on each spacelike hypersurface. The conditions then tell us the limiting behaviour of the metric as the distance from the sources tends to infinity. Very often, however, we need to know how the metric at a *given* point depends on the nature of the sources. We shall usually consider a family of spacetimes labelled by a real parameter M, which we may loosely regard as the total mass of the sources, and we assume that $h_{\mu\nu}(x) = O(M)$ as $M \to 0$ for each fixed x. (We shall write this simply as $h_{\mu\nu} = O(M)$). It follows that $h_{\mu\nu,\lambda} = O(M)$, and similarly for the higher derivatives. We extend the notation in the obvious way: if $f = O(L^p)$, $g = O(M^n)$, then $f+g = O(L^p)+O(M^n)$, $fg = O(L^pM^n)$, etc.

Before continuing the discussion of generalized Minkowski charts, we recall some definitions and elementary results of special relativity. The 4×4 matrix (L^ν) is a *Lorentz matrix* if $L^\nu_\mu \in R^1$ and $\eta_{\mu\nu}L^\nu L^\nu_\rho = \eta_{\pi\rho}$. The Lorentz matrix (L^ν) is a *restricted* Lorentz matrix if $\det(L^\nu) = 1$ and $L_0 > 0$; it is a Lorentz matrix *without rotation of the spatial axes* if $L^\nu = L^\mu_\nu$. Any restricted Lorentz matrix without rotation of the spatial axes can be written $(L^\nu_\mu(\boldsymbol{\beta}))$, where

$$\left.\begin{aligned} L^0_0(\boldsymbol{\beta}) = \gamma, \qquad L^0_m(\boldsymbol{\beta}) = L^m_0(\boldsymbol{\beta}) = -\gamma\beta_m \\ L^n_m(\boldsymbol{\beta}) = \delta_{mn}+(\gamma-1)\beta_m\beta_n\beta^{-2} \end{aligned}\right\} \tag{2.2}$$

and

$$\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3) \in R^3, \qquad \beta = (\beta_m\beta_m)^{1/2} < 1, \qquad \gamma = (1-\beta^2)^{-1/2}.$$

If S and S' are inertial charts with coordinates x and x' respectively, c_{E} is the speed of light, and $x'^\mu = L_\mu(\boldsymbol{\beta})x^\nu$, then $c_{\mathrm{E}}\boldsymbol{\beta}$ is the velocity in S of a particle which is stationary in S'.

The set of generalized Minkowski charts in the spacetime Σ is much larger than the set of Minkowski charts in flat spacetime. Fortunately, there is no need for us to determine the whole set (for discussion of a similar problem see Bondi *et al.* 1962, and Sachs 1962). We require only a subset $\Lambda(m)$ of the set of generalized Minkowski charts, where $\Lambda(m)$ is in one-to-one correspondence with the set of real three-vectors $\boldsymbol{\beta}$ that satisfy $\beta < 1$, and m is a set of parameters that determine the material sources of the gravitational field. We suppose that these sources can be regarded as a superposition of a denumerable set of mass distributions, each of which is stationary in one of the charts of $\Lambda(m)$. We take m_A to be the total proper mass which is stationary in $S_A \in \Lambda(m)$ for $A = 0, 1, 2, \ldots$. For simplicity we consider a set of mass distributions that are entirely determined by the m_A, so that we may put $m = (m_0, m_1, m_2, \ldots)$.

Under the one-to-one correspondence between $\Lambda(m)$ and $\{\boldsymbol{\beta} \in R^3 | \beta < 1\}$, we take S_A to correspond to $\boldsymbol{\beta}_A$, and $\boldsymbol{\beta}_0 = 0$. If the coordinates of S_A are x_A, we assume that

$$x_A^{\mu} = L^{\mu}_{\nu}(\boldsymbol{\beta}_A)x_0^{\nu} + f^{\mu}(x_0, \boldsymbol{\beta}_A, m) \tag{2.3}$$

where $f^{\mu}(x_0, 0, m) = 0$, $f^{\mu}_{,\nu} = \partial f^{\mu}/\partial x_0^{\nu} = O(L^{-2})$ for fixed $\boldsymbol{\beta}_A$ and m, and $f^{\mu} = O(M)$ for fixed x_0 and $\boldsymbol{\beta}_A$, with $M = \Sigma^{\infty}{}_{=0} m_B$. We call $V_A = c_E \boldsymbol{\beta}_A$ the *asymptotic velocity* of S_A in S_0. For small f^{μ} it is approximately the velocity in S_0 of a particle fixed in S_A.

The components of any tensor (or tensor field) t in S_A will be written $t^{A\mu\nu\cdots}_{\lambda\pi\cdots}$. It is assumed that in any chart S_A the components of the metric satisfy

$$\overset{A}{g}_{\mu\nu} = \eta_{\mu\nu} + \sum_{B=0}^{\infty} m_B \overset{A}{g}_{B\mu\nu} + O(M^2) + O(L^{-2}) \tag{2.4}$$

where the $g^A_{B\mu\nu}$ are independent of the m_B and $g^A_{B\mu\nu} = O(L^{-1})$, and where the limit $M \to 0$ is to be taken in such a manner that the ratios m_A/M stay constant. It follows from (2.3) that if (2.4) holds for any $S_A \in \Lambda(m)$, then it holds for all $S_A \in \Lambda(m)$.

If $m_B = 0$ for $B \neq A$, then (2.4) becomes $g_{\mu\nu} = \eta_{\mu\nu} + m_A g_{A\mu\nu} + O(M^2) + O(L^{-2})$. It is assumed that the $g^A_{A\mu\nu}$ are known: that is, we know how to solve the problems of gravitostatics to first order in m_A in any chart $S_A \in \Lambda(m)$. Using (2.3) again, one has

$$\overset{0}{g}_{B\mu\nu} = L^{\pi}_{\mu}(\boldsymbol{\beta}_B) L^{\rho}_{\nu}(\boldsymbol{\beta}_B) \overset{B}{g}_{B\pi\rho} + O(M^2) + O(L^{-2})$$

and substituting in (2.4) gives

$$g^0_{\mu\nu} = \eta_{\mu\nu} + \sum_{B=0}^{\infty} m_B L^{\pi}_{\mu}(\boldsymbol{\beta}_B) L^{\rho}_{\nu}(\boldsymbol{\beta}_B) \overset{B}{g}_{B\pi\rho} + O(M^2) + O(L^{-2}). \tag{2.5}$$

In the theories of gravitation that are usually considered, the terms in the expression for $g^0_{\mu\nu}$ that are $O(M^2)$ are also $O(L^{-2})$, and one can therefore replace the last two terms in equation (2.5) by $O(M^2L^{-2})$.

Before one can use (2.5) to calculate the metric due to a moving body, one must specify the functions $g^A_{A\mu\nu}$. We assume that

$$\left.\begin{aligned} \overset{A}{g}_{Amn}(x_A) &= \delta_{mn} p r_A^{-1} + O(L^{-2}) \\ \overset{A}{g}_{A\mu 0}(x_A) &= \overset{A}{g}_{A0\mu}(x_A) = \delta_{\mu 0} q r_A^{-1} + O(L^{-2}) \end{aligned}\right\} \tag{2.6}$$

where $r_A = [(x_A^k - a_A^k)(x_A^k - a_A^k)]^{1/2}$, and p, q, a_A^k are constants. It follows that, in the special case when $m_B = 0$ for $B \neq A$, equation (2.4) becomes

$$\left.\begin{aligned} \overset{A}{g}_{mn}(x_A) &= \delta_{mn}(1 + m_A p r_A^{-1}) + O(M^2) + O(L^{-2}) \\ \overset{A}{g}_{\mu 0}(x_A) &= \delta_{\mu 0}(-1 + m_A q r_A^{-1}) + O(M^2) + O(L^{-2}). \end{aligned}\right\} \tag{2.7}$$

One can regard (2.7) as giving the metric due to a particle of proper mass m_A fixed in S_A at the point with spatial coordinates a_A^k. Thus in assuming (2.6) we are taking the mass distribution to consist of a set of particles, each stationary in one of the generalized Minkowski charts of $\Lambda(m)$. In order that the theory should reduce to Newton's in the appropriate limit, one must have $q = 2G_E c_E^{-2}$, where G_E is the Newtonian gravitational constant.

An important physical assumption which is contained in (2.6) is that the constants p and q are the same for all the charts of $\Lambda(m)$. This assumption, which is related to Mach's principle, is our generalization of the special-relativistic hypothesis that all Minkowski charts are equivalent. Intuitively speaking, it means that the metric produced in an initially Minkowski chart S by putting a stationary particle of given proper mass into it is independent of how one chooses S.

To find the metric of a system of particles in the generalized Minkowski chart S_0, one has only to substitute (2.6) in (2.5). Using (2.2) and the equation $\eta_{\mu\nu}L^\mu_\pi L^\nu_\rho = \eta_{\pi\rho}$, we find

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \sum_{A=0}^{\infty} m_A r_A^{-1}[p\eta_{\mu\nu} + (p+q)\beta_{A\mu}\beta_{A\nu}\gamma_A{}^2] + O(M^2) + O(L^{-2}) \qquad (2.8)$$

where $\beta_{A0} = -1$, and where we have simplified the notation by writing x in place of x_0 and $g_{\mu\nu}$ in place of $g^0_{\mu\nu}$.

We must express r_A in terms of the S_0 coordinates. Let $a_A^0 \in R^1$, and let a^μ_{A0}, x^μ be the S_0 coordinates of the points whose S_A coordinates are a^μ_A and x^μ_A respectively. Define $r^m_{A0} = x^m - a^m_{A0}$, $r_{A0} = (r^m_{A0} r^m_{A0})^{1/2} = |\boldsymbol{x} - \boldsymbol{a}_{A0}|$, and $a^0_{A0} = x^0 - r_{A0}$. The constant a^0_A is so far arbitrary. We now choose it so that $\eta_{\mu\nu}(x^\mu - a^\mu_{A0})(x^\nu - a^\nu_{A0}) = 0$, $x^0 > a^0_{A0}$. Using (2.3) and the conditions imposed on the f^μ, we find that

$$r_A^{-1} = [\gamma_A(r_{A0} - \beta_{Am} r^m_{A0})]^{-1} + O(ML^{-2}). \qquad (2.9)$$

The equations of the world line of particle A in S_0 can be written $z^m_{A0} = \pi^m(u)$, $z^0_{A0} = u$, where $u \in R^1$. We regard particle A as corresponding to a proper mass density ρ_A in S_0, given by

$$\left.\begin{aligned} \rho_A(y) = \rho_A(y^0, \boldsymbol{y}) &= m_A \delta^3(\boldsymbol{y} - \boldsymbol{\pi}(y^0)) \\ &= m_A \int_{-\infty}^{\infty} \delta^3(\boldsymbol{y} - \boldsymbol{\pi}(u))\delta(y^0 - u)\,\mathrm{d}u. \end{aligned}\right\} \qquad (2.10)$$

If $G: R^4 \to R^1$ is a smooth function, then a short calculation (cf. the derivation of the Liénard–Wiechert potentials in electromagnetism) shows that

$$\left.\begin{aligned} &\int \rho_A(x^0 - R, \boldsymbol{y}) R^{-1} G(x^0 - R, \boldsymbol{y})\,\mathrm{d}^3 y \\ &\quad = m_A G(u_0, \boldsymbol{\pi}(u_0))[x^0 - u_0 + \pi^{m\prime}(u_0)\,(\pi^m(u_0) - x^m)]^{-1} \end{aligned}\right\} \qquad (2.11)$$

where the integral is over all $(y^1, y^2, y^3) \in R^3$, $R = |\boldsymbol{x} - \boldsymbol{y}| = [(x^m - y^m)(x^m - y^m)]^{1/2}$, and u_0 is defined by $u_0 + |\boldsymbol{x} - \boldsymbol{\pi}(u_0)| = x^0$. It follows from the previous definitions that $u_0 = a^0_{A0}$, $\pi^m(u_0) = a^m_{A0}$, $\pi^{m\prime}(u_0) = V_{Am} c_E^{-1} = \beta_{Am}$, and from (2.9) that the right-hand side of (2.11) is

$$m_A G(x^0 - r_{A0}, \boldsymbol{x} - \boldsymbol{r}_{A0}) \gamma_A (r_A^{-1} + O(ML^{-2})).$$

If we define $c_E \boldsymbol{\beta}(y)$ to be the three-velocity of the mass distribution in S_0 at y, and $\gamma(y) = (1-\beta^2(y))^{-1/2}$, then $\boldsymbol{\beta}_A = \boldsymbol{\beta}(x^0 - r_{A0},\ \boldsymbol{x} - \boldsymbol{r}_{A0})$, $\gamma_A = \gamma(x^0 - r_{A0},\ \boldsymbol{x} - \boldsymbol{r}_{A0})$, and (2.8) becomes

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \int [\rho(y) R^{-1} \gamma^{-1}(y)[p\eta_{\mu\nu} + (p+q)\beta_\mu(y)\beta_\nu(y)\gamma^2(y)]]_{y^0 = x^0 - R}\, d^3y$$
$$+ O(M^2 L^{-2}) \tag{2.12}$$

where $\rho = \Sigma_{A=0}^{\infty} \rho_A$ is the S_0 proper mass density (the total proper mass per unit three-volume in S_0), and where we assume that the terms $O(L^{-2})$ in (2.8) are also $O(M^2)$.

One shows by the usual limiting procedure that (2.12) also gives the metric due to a continuous mass distribution. In this case ρ and β are continuous functions which we shall take to be piecewise smooth and of bounded support. The field equation for $g_{\mu\nu}$ that follows from (2.12) is

$$\eta_{\pi\rho} g_{\mu\nu,\pi\rho} = -4\pi\rho\gamma^{-1}[p\eta_{\mu\nu} + (p+q)\beta_\mu\beta_\nu\gamma^2] + O(M^2) \tag{2.13}$$

where the comma denotes the partial derivative with respect to the x^μ.

Define the *proper mass density* ρ_{pr} by $\rho_{pr} = \rho\gamma^{-1}$ (this is an invariant quantity). The covariant components of the four-velocity of the mass distribution are $u_\mu = \gamma\beta_\mu c_E + O(M)$, because $g_{\mu\nu} = \eta_{\mu\nu} + O(M)$. The covariant components of the energy-momentum density of the mass distribution are $T_{\mu\nu} = \rho_{pr} u_\mu u_\nu = O(M)$, and hence $\rho_{pr} = c_E^{-2} \eta_{\mu\nu} T_{\mu\nu} + O(M^2)$. We can now rewrite (2.12) and (2.13) as

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + c_E^{-2} \int R^{-1}[-p\eta_{\pi\rho} T_{\pi\rho}(y)\,\eta_{\mu\nu} + (p+q) T_{\mu\nu}(y)]_{y^0 = x^0 - R}\, d^3y + O(M^2) \tag{2.14}$$

$$\eta_{\pi\rho} g_{\mu\nu,\pi\rho} = -4\pi c_E^{-2}[-p\eta_{\pi\rho} T_{\pi\rho}\eta_{\mu\nu} + (p+q) T_{\mu\nu}] + O(M^2). \tag{2.15}$$

One proves easily from (2.14) that

$$\eta_{\nu\pi} g_{\mu\nu,\pi}(x) - \tfrac{1}{2}\eta_{\nu\pi} g_{\nu\pi,\mu}(x)$$
$$= c_E^{-2} \int R^{-1}[\tfrac{1}{2}(p-q)\eta_{\nu\pi} T_{\nu\pi,\mu}(y) + (p+q)\eta_{\nu\pi} T_{\mu\nu,\pi}(y)]_{y^0 = x^0 - R}\, d^3y + O(M^2). \tag{2.16}$$

This result will be needed in § 4.

3. Lense–Thirring effect

We have calculated the approximate form of the metric for bodies that move at arbitrary speed. However, in many problems the speeds of the bodies are much less than the speed of light, and the equations of § 2 can be considerably simplified. As an illustration, we calculate the metric due to a slowly rotating, spherically symmetric mass distribution. The effects of acceleration are again neglected.

If in (2.15) (or (2.12)) one drops all terms that are of at least second degree in the velocity components V_m, one finds

$$g_{\mu\nu}(x) \simeq \eta_{\mu\nu} + \int [\rho(y) R^{-1}[p\eta_{\mu\nu} + (p+q)\delta_{\mu 0}\delta_{\nu 0} - (p+q) c_E^{-1}\, V_m(y)$$
$$\times (\delta_{\mu m}\delta_{\nu 0} + \delta_{\nu m}\delta_{\mu 0})]]_{y^0 = x^0 - R}\, d^3y. \tag{3.1}$$

We shall use (3.1) to calculate the components g_{m0} of the metric of a spherically symmetric mass distribution that rotates rigidly with a constant angular velocity $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ about an axis of symmetry. The centre of the mass distribution is fixed at the spatial origin, so that its velocity at x is $V(x^0, \boldsymbol{x}) = \boldsymbol{\omega} \times \boldsymbol{x}$. It follows that V_m and ρ in (3.1) are independent of y^0.

Define $r = |\boldsymbol{x}| = (x^m x^m)^{1/2}$, $s = |\boldsymbol{y}| = (y^m y^m)^{1/2}$. For fixed s one has $R^{-1} = |\boldsymbol{x}-\boldsymbol{y}|^{-1} = r^{-1}(1+r^{-1}s\cos\theta + O(r^{-2}s^2))$ as $r \to \infty$, where θ is the angle between $\boldsymbol{x}$ and $\boldsymbol{y}$, and hence

$$\int \rho(y)\, V(y) R^{-1}\, \mathrm{d}^3 y \simeq r^{-1}\boldsymbol{\omega} \times \int \rho(y)\, \boldsymbol{y}\, \mathrm{d}^3 y + r^{-2}\boldsymbol{\omega} \times \int \rho(y) s \boldsymbol{y} \cos\theta\, \mathrm{d}^3 y \tag{3.2}$$

where the $O(r^{-2}s^2)$ term has been neglected. The first term of (3.2) vanishes because of the spherical symmetry, and the second is $\frac{1}{2} I r^{-3} \boldsymbol{\omega} \times \boldsymbol{x}$, where $I = \int \rho(y) s^2 \sin^2\theta\, \mathrm{d}^3 y$ is the moment of inertia of the mass distribution about an axis of symmetry. Substituting in (3.1) gives

$$g_{m0}(x) \simeq -\frac{(p+q)I}{2c_{\mathrm{E}} r^3} \epsilon_{mnp} \omega_n x^p. \tag{3.3}$$

The constants p and q were defined by (2.6). As explained in connection with equation (2.7), we must take $q = 2G_{\mathrm{E}} c_{\mathrm{E}}^{-2}$. The deflection of starlight by the Sun and the radar time-delay of Shapiro (1964) are both compatible with $p = 2G_{\mathrm{E}} c_{\mathrm{E}}^{-2}$ (Shapiro *et al.* 1968). With this choice, and with $\omega_n = \omega \delta_{3n}$, equation (3.3) becomes

$$g_{10}(x) \simeq \frac{2G_{\mathrm{E}} I \omega x^2}{c_{\mathrm{E}}^3 r^3}, \qquad g_{20}(x) \simeq \frac{-2G_{\mathrm{E}} I \omega x^1}{c_{\mathrm{E}}^3 r^3}, \qquad g_{30}(x) \simeq 0. \tag{3.4}$$

These are expressions for the non-diagonal components of the metric of a rotating body that were first derived by Lense and Thirring (1918), who used the linearized form of the Einstein field equations (see Landau and Lifshitz 1962—§ 103). We emphasize again that we have not assumed any field equations. Attempts are being made to test (3.4) by measuring the precession of a spinning satellite (Schiff 1960, Rastall 1966, Cooper *et al.* 1968).

4. Gravitational field equations

We have shown that, for an unaccelerated matter distribution, the components of the metric satisfy the approximate gravitational field equations (2.15) and the subsidiary conditions (2.16). If $p = q = 2G_{\mathrm{E}} c_{\mathrm{E}}^{-2}$, as assumed at the end of the last section, and if $\eta_{\nu\pi} T_{\mu\nu,\pi} = O(M^2)$ (recall that $\eta_{\nu\pi} T_{\mu\nu,\pi} = 0$ in the Minkowski charts of special relativity, and that $T_{\mu\nu} = O(M)$) then (2.16) becomes

$$\eta_{\nu\pi} g_{\mu\nu,\pi} - \tfrac{1}{2}\eta_{\nu\pi} g_{\nu\pi,\mu} = O(M^2). \tag{4.1}$$

Since $g_{\mu\nu} = \eta_{\mu\nu} + O(M)$, we have $g^{\mu\nu} = \eta_{\mu\nu} + O(M)$, the Christoffel symbols are $O(M)$, and the components of the Ricci tensor are given by

$$\begin{aligned}\mathscr{R}_{\tau\pi} &= \tfrac{1}{2}\eta_{\sigma\nu}(g_{\sigma\nu,\tau\pi} + g_{\tau\pi,\sigma\nu} - g_{\sigma\pi,\tau\nu} - g_{\tau\nu,\sigma\pi}) + O(M^2) \\ &= \tfrac{1}{2}\eta_{\sigma\nu} g_{\tau\pi,\sigma\nu} - \tfrac{1}{2}(\eta_{\sigma\nu} g_{\pi\sigma,\nu} - \tfrac{1}{2}\eta_{\sigma\nu} g_{\sigma\nu,\pi})_{,\tau} \\ &\quad - \tfrac{1}{2}(\eta_{\sigma\nu} g_{\tau\nu,\sigma} - \tfrac{1}{2}\eta_{\sigma\nu} g_{\sigma\nu,\tau})_{,\pi} + O(M^2).\end{aligned} \tag{4.2}$$

From (4.1) and (4.2) we find $R_{\mu\nu} = \frac{1}{2}\eta_{\sigma\tau} g_{\mu\nu,\sigma\tau} + O(M^2)$, so the field equations (2.15) can be written

$$\mathscr{R}_{\mu\nu} = -\kappa(T_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}g^{\pi\rho}T_{\pi\rho}) + O(M^2) \tag{4.3}$$

where $\kappa = 8\pi G_{\mathrm{E}}c_{\mathrm{E}}^{-4}$. The curvature scalar $\mathscr{R}$ is defined by $\mathscr{R} = g^{\mu\nu}\mathscr{R}_{\mu\nu}$. From (4.3), $\mathscr{R} = \kappa g^{\pi\rho}T_{\pi\rho} + O(M^2)$, and (4.3) has the alternative form

$$\mathscr{R}_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}\mathscr{R} = -\kappa T_{\mu\nu} + O(M^2). \tag{4.4}$$

If one omits the terms $O(M^2)$ (including those in $\mathscr{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathscr{R}$), then (4.4) become the linearized Einstein field equations. In this approximate sense our metric is a solution of the linearized Einstein field equations.

5. Conclusions

The results that we have derived enable us to understand the relations between three different effects: the precession of the axis of a spinning satellite, the deflection of starlight by the Sun, and the time delay in the Shapiro radar experiment (Shapiro 1964). We have shown that if the metric due to a particle fixed in a generalized Minkowski chart S_A is given by (2.7), then the non-diagonal components of the metric of a spherically symmetric, slowly rotating body are proportional to the constant $p+q$ (equation (3.3)). The precession of the axis of a spinning satellite then includes a term proportional to $p+q$.

One can prove that the other two effects are determined by the same parameter $p+q$. To make this plausible, let the equations of a light ray in S_A be $z_A^\mu = f^\mu(u)$, where $f^0(u) > 0$ for all u. Define the *speed of light in* S_A to be

$$c = c_{\mathrm{E}}(f^{m\prime}(u)f^{m\prime}(u))^{1/2}/f^{0\prime}(u)$$

where c_{E} is the speed of light (a universal constant). Since the tangent to a light ray is a null vector, equation (2.7) implies that

$$c = c_{\mathrm{E}}[1 - \tfrac{1}{2}(p+q)r_A^{-1}m_A] + O(L^{-2}) + O(M^2).$$

Thus light behaves in S_A as though space had a refractive index

$$c_{\mathrm{E}}/c \simeq 1 + \tfrac{1}{2}(p+q)r_A^{-1}m_A.$$

Because the metric due to the Sun is asymptotically of the form (2.7), one can take this to be the refractive index of space in the solar system. By simple arguments of geometrical optics, one now shows that both the light deflection and the time delay are determined by the refractive index, and hence by $p+q$. (For a more complete argument see Dyson 1967).

The constant q is known, so the experiments will allow us to measure p in three independent ways. If the results are inconsistent, it may mean that the correct theory of gravitation is not of the traditional Riemannian type. On the other hand, it may only be that one of our assumptions about asymptotic behaviour is too strong (e.g. it is possible that $f^\mu_{,\nu}$ in equation (2.3) is not $O(L^{-2})$).

References

Bondi, H., van der Burg, M. G. H., and Metzner, A. W. K., 1962, *Proc. R. Soc.* A, **269**, 21.
Cooper, D. H., Karr, G. R., Myers, J. L., and Skaperdas, D., 1968, Rep. R-378 (Urbana, Illinois: University of Illinois Coordinated Science Laboratory).

DICKE, R. H., and GOLDENBERG, H. M., 1967, *Phys. Rev. Lett.*, **18**, 313.
DYSON, F. J., 1967, in *Lectures in Applied Mathematics*, Vol. 8, Ed. J. Ehlers (Providence, Rhode Island: American Mathematical Society).
LANDAU, L. D., and LIFSHITZ, E. M., 1962, *The Classical Theory of Fields*, 2*nd* edn (Oxford: Pergamon).
LENSE, J., and THIRRING, H., 1918, *Phys. Z.*, **19**, 156.
O'HANLON, J., RASTALL, P., and TAM, K. K., 1969, *J. Phys.* A: *Gen. Phys.*, **2**, 141.
RASTALL, P., 1966, *Can. J. Phys.*, **44**, 3109.
—— 1968 a, *Can. J. Phys.*, **46**, 2155.
—— 1968 b, *J. Phys.* A: *Gen. Phys.*, **1**, 501.
—— 1969, *Can. J. Phys.*, **47**, 2161.
SACHS, R. K., 1962, *Proc. R. Soc.*, A, **270**, 103.
SCHIFF, L. I., 1960, *Proc. Natn. Acad. Sci., U.S.A.*, **46**, 871.
SHAPIRO, I. I., 1964, *Phys. Rev. Lett.*, **13**, 789.
SHAPIRO, I. I., *et al.*, 1968, *Phys. Rev. Lett.*, **20**, 1265.
TOLMAN, R. C., 1934, *Relativity, Thermodynamics, and Cosmology* (London: Oxford University).